

3. Frequency domain

Signal and sample series. Fourier transform. Frequency spectrum. Fourier space filtering. Convolution theorem. Application examples.

Signal and sample series

An elemental periodic wave such as sine is defined by one of the following expressions (as a function of period T , or frequency f):

$$P(t) = A \times \sin\left(\frac{2\pi}{T}t + \phi\right) + C$$

$$P(t) = A \times \sin(2\pi ft + \phi) + C$$

A : amplitude; f : frequency; T : period; ϕ : phase; C : translation

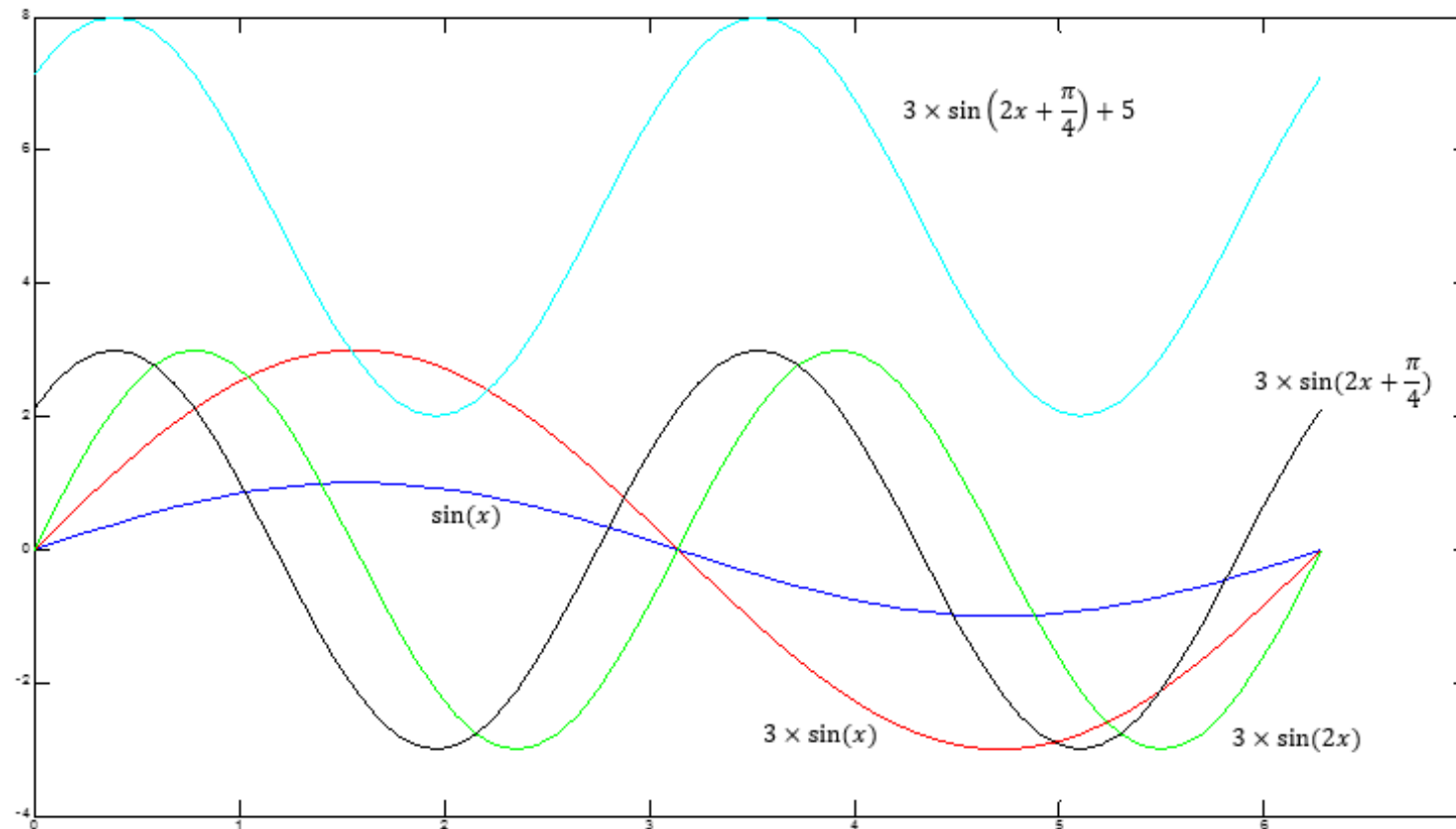


Signal and sample series

Periodic wave examples

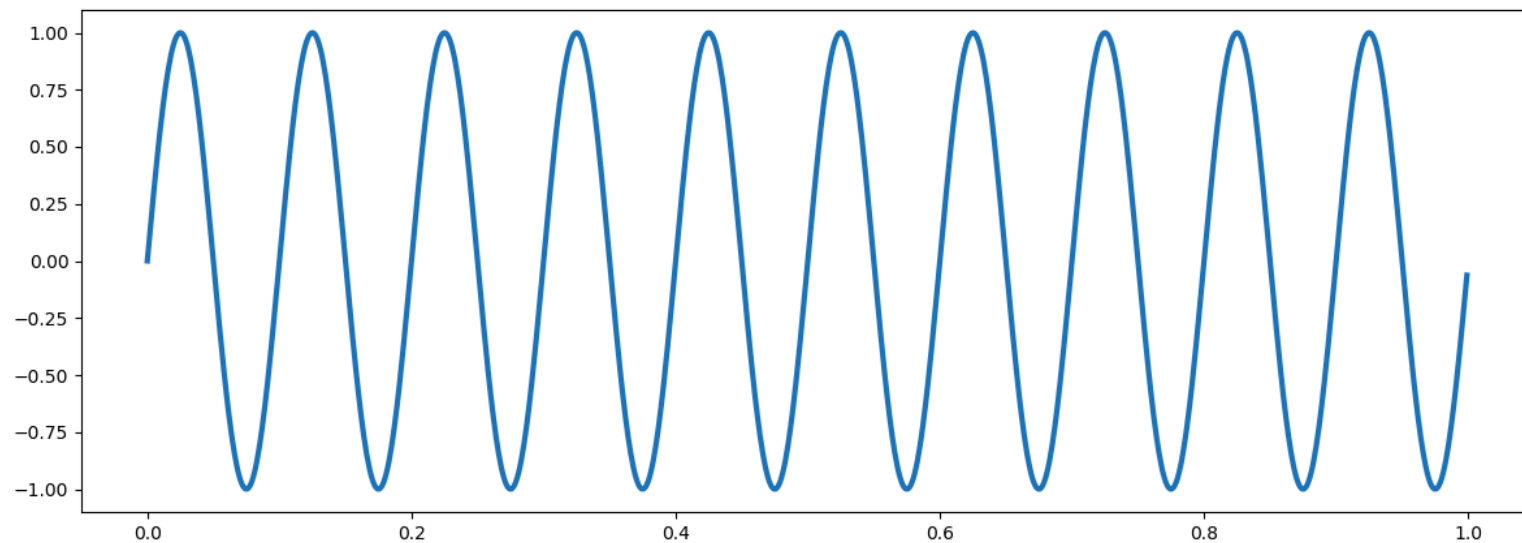
$$y = a \times \sin(\omega x + \varphi) + d$$

$$\omega = 2\pi f$$



Signal and sample series

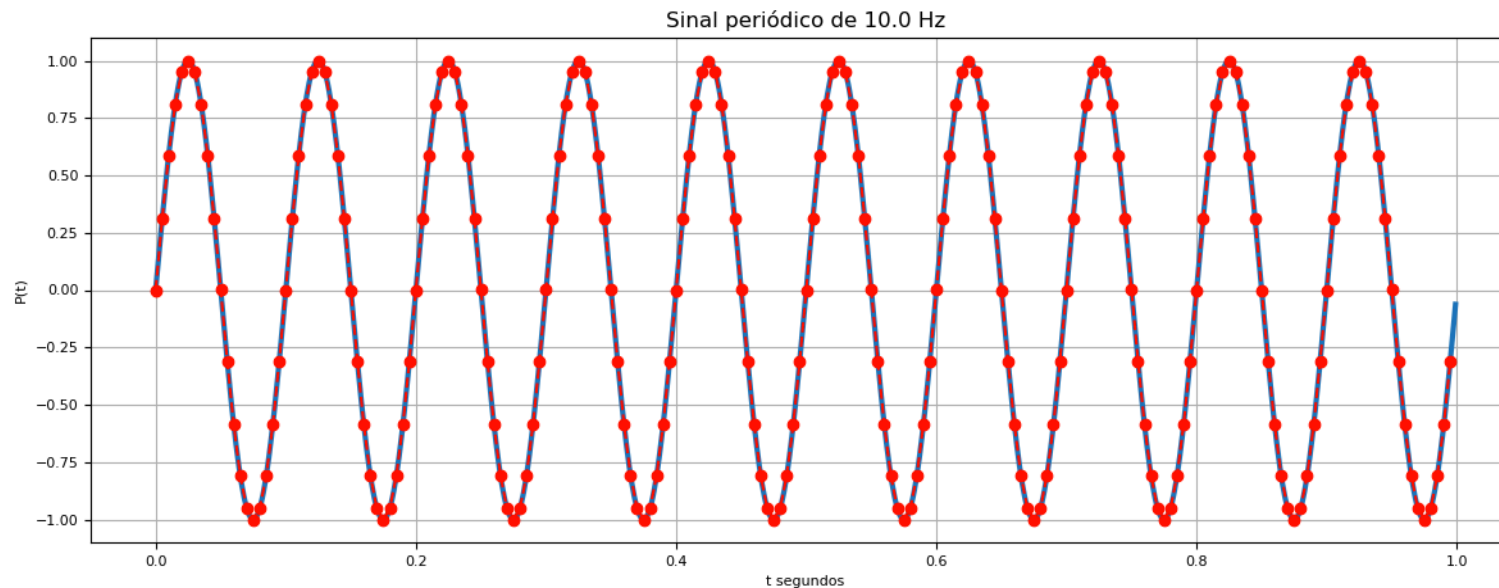
In a periodic signal P , the frequency value f corresponds to the number of cycles m occurring in the time interval of one second (or unit of time) given in Hertz (Hz).



Signal with a frequency of 10 Hz

Signal and sample series

In a process of digitizing an analog signal, P sampling consists of generating a discrete pulse sequence S (sample series) with a sampling frequency f_s .



Signal and sample series

The Nyquist-Shannon sampling theorem establishes the condition that determines the smallest sampling frequency that makes signal reconstruction appropriate:

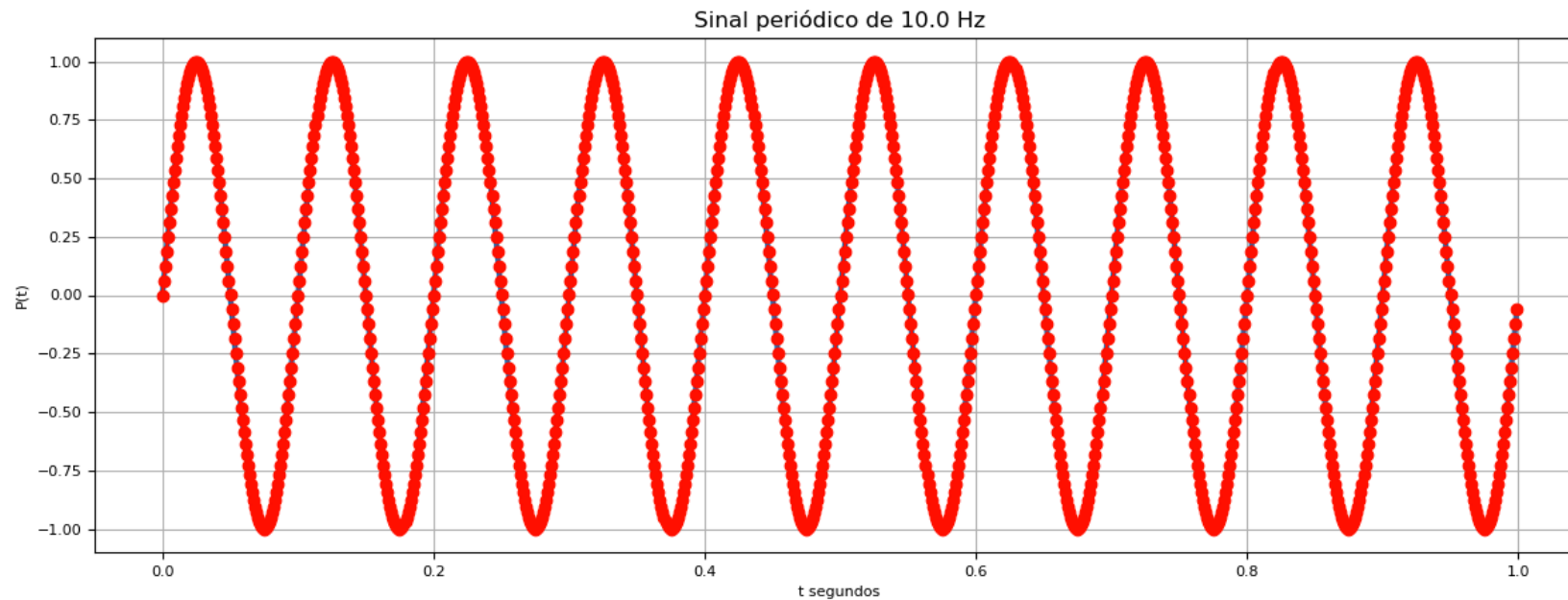
$$f_s > 2 \times f_{max}$$

In other words, the sampling frequency must be greater than twice the maximum signal frequency.

Signal and sample series

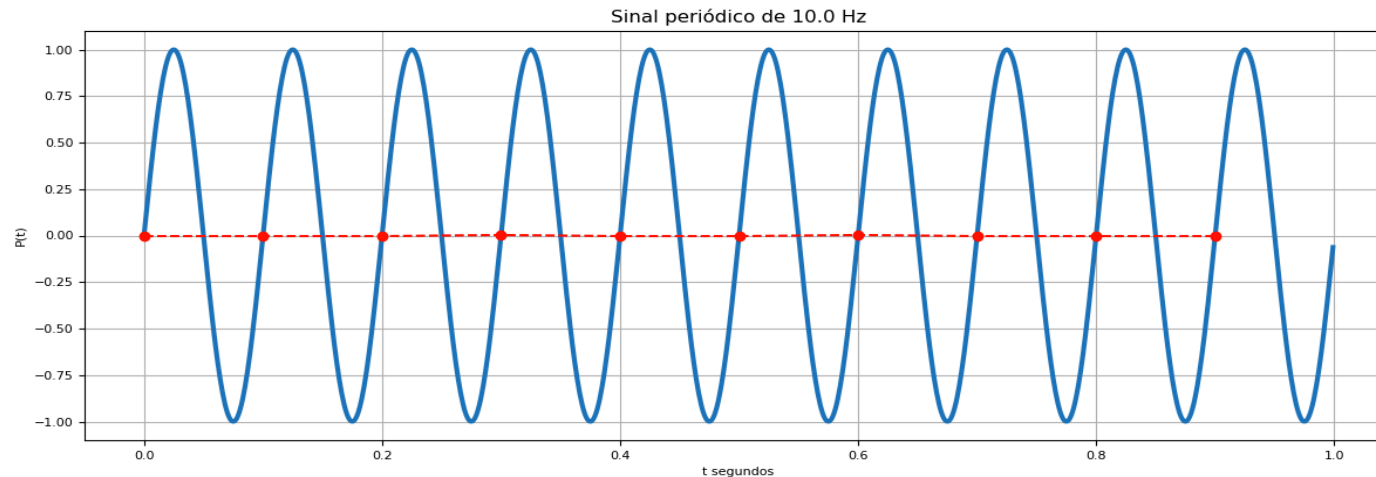
In this perspective, the following question may be asked:

- Why not then sample the signal with the highest possible value for f , or always a high value?



Signal and sample series

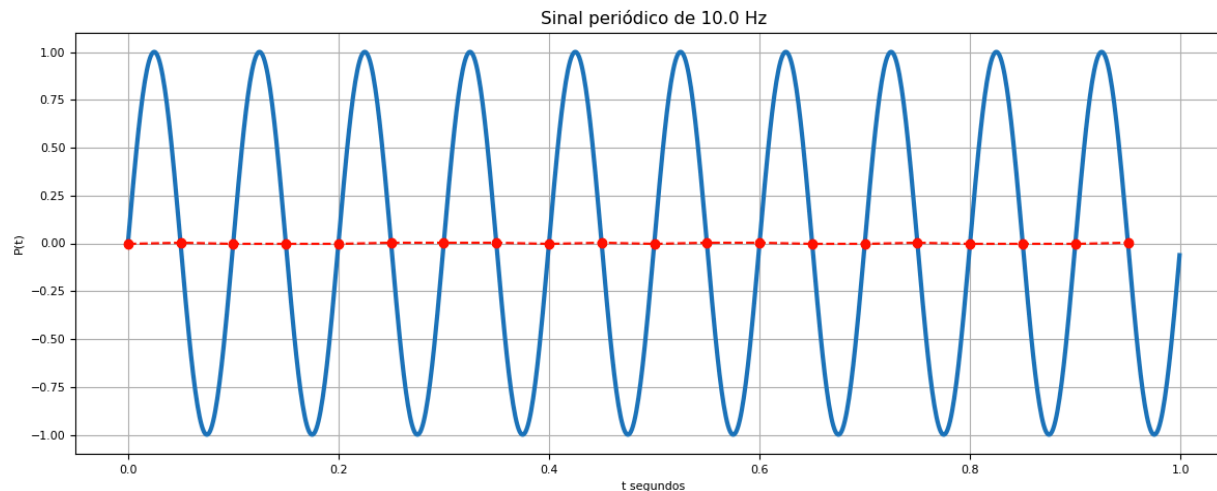
Consider, for example, a periodic P wave of a sine with $f = 10$ Hz, i.e. a signal with a frequency of ten cycles per second (simplified with $A = 1$ and $\phi = C = 0$). A series with $f_s = 10$ Hz (which does not meet the above condition) has one sample per period.



Connecting these points results in a line that has no relation to the shape of the signal.

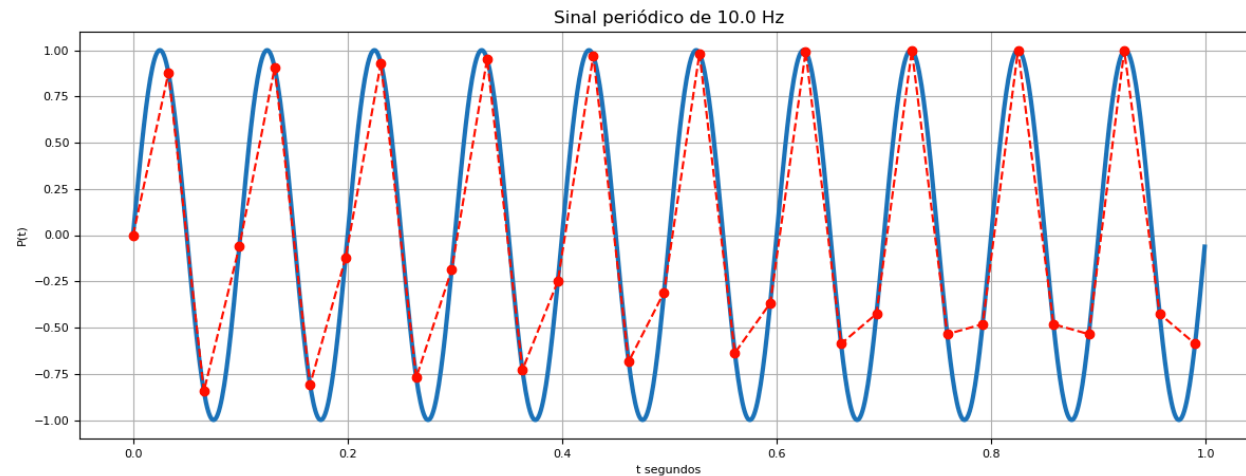
Signal and sample series

Consider now a series with $f_s = 210 = 20$ Hz, which is at the limit of Nyquist condition (but not yet respecting it); In this case there are two samples per period which, connected, form a line similar to the previous one.



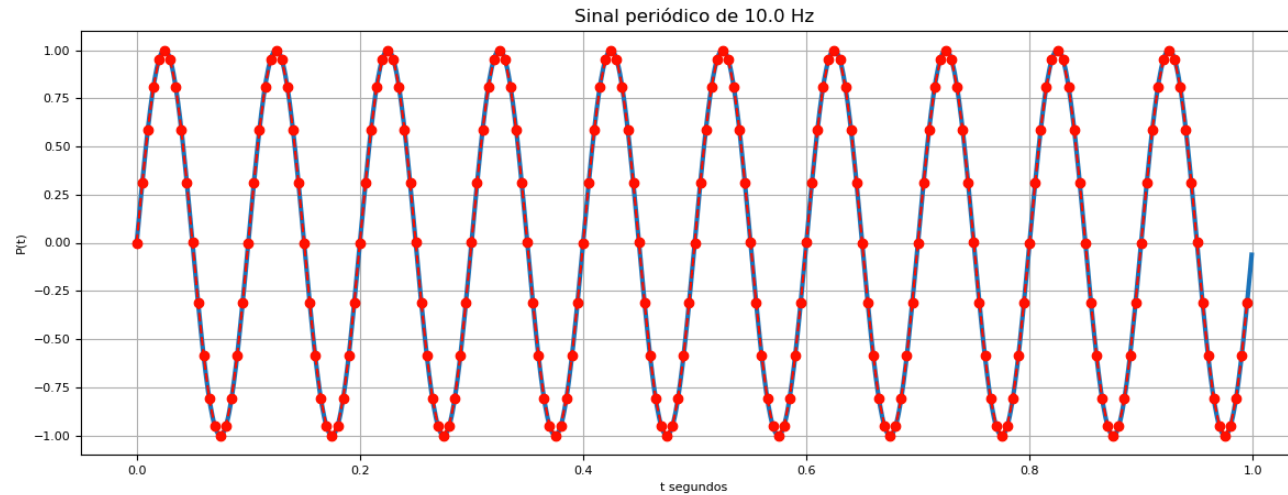
Signal and sample series

By increasing the sampling frequency to a value that meets the condition, for example, $f_s = 310 = 30$ Hz, it can be seen that the series behaves in such a way that the signal can already be clearly reconstructed and may be almost sufficient to the goals to be met.



Signal and sample series

With a higher value of, for example, $f_s = 2010 = 200 \text{ Hz}$, we have a sampled series already quite faithful to the original signal.



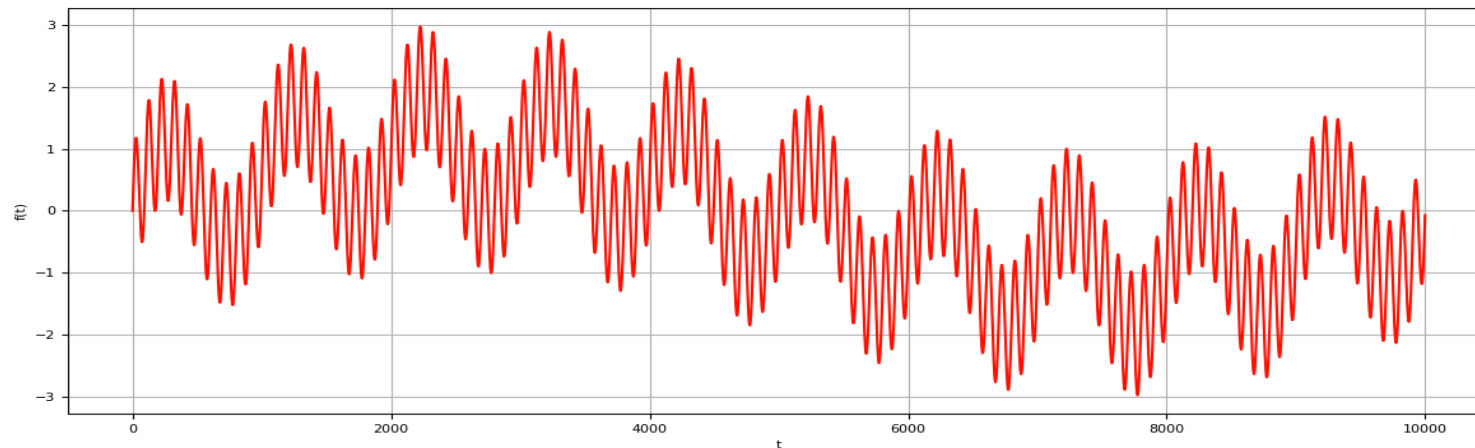


Signal and sample series

Therefore, for a type of signal with a known bandwidth, it is unnecessary to sample it with sampling frequencies above those which allow unambiguous reconstitution.

Signal and sample series

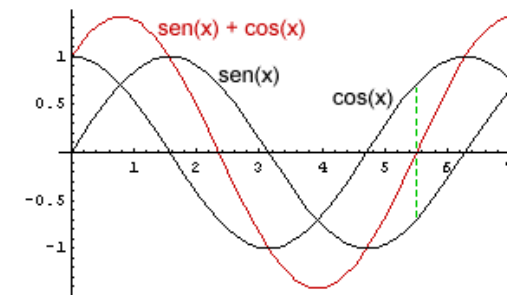
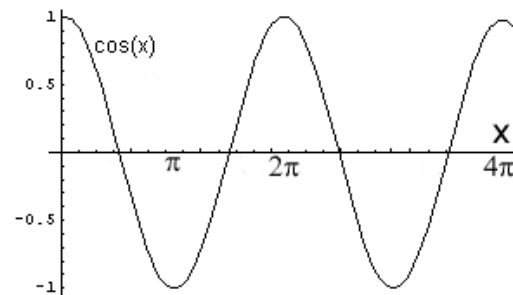
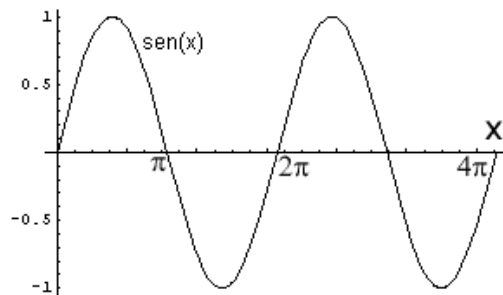
Consider a second synthetic example, produced from the sum of three sine waves with different frequencies not known at the outset and which, although equally periodic, has a more difficult behavior to interpret. The goal will be to know the characteristics of the elementary waves. For this we use the spectral analysis using the **Fourier transform**.



Fourier Transform

Fourier's theory states that any signal can be expressed as a sum of several sine waves, that is, a sum of sine and cosine functions.

$$f(x) = a_0 + a_1 \text{sen}(x) + a_2 \text{sen}(2x) + a_3 \text{sen}(3x) + \dots + b_1 \text{cos}(x) + b_2 \text{cos}(2x) + b_3 \text{cos}(3x) + \dots$$



The Fourier transform (TF) allows the decomposition of a signal (defined in the spatial domain) into its sine and cosine components, representing it in the so-called “frequency domain”, or Fourier space.

1D Fourier Transform

Direct Transform:

$$F(u) = \sum_{x=0}^{N-1} \left[f(x) \times e^{-j \frac{2\pi u x}{N}} \right] = \sum_{x=0}^{N-1} \left[f(x) \times \left(\cos \left(-\frac{2\pi u x}{N} \right) + j \sin \left(-\frac{2\pi u x}{N} \right) \right) \right]$$

Inverse Transform:

$$f(x) = \frac{1}{N} \times \sum_{u=0}^{N-1} \left[F(u) \times e^{j \frac{2\pi u x}{N}} \right] = \frac{1}{N} \times \sum_{u=0}^{N-1} \left[f(x) \times \left(\cos \left(\frac{2\pi u x}{N} \right) + j \sin \left(\frac{2\pi u x}{N} \right) \right) \right]$$

N = number of samples

x = current sample ($0 \dots N-1$)

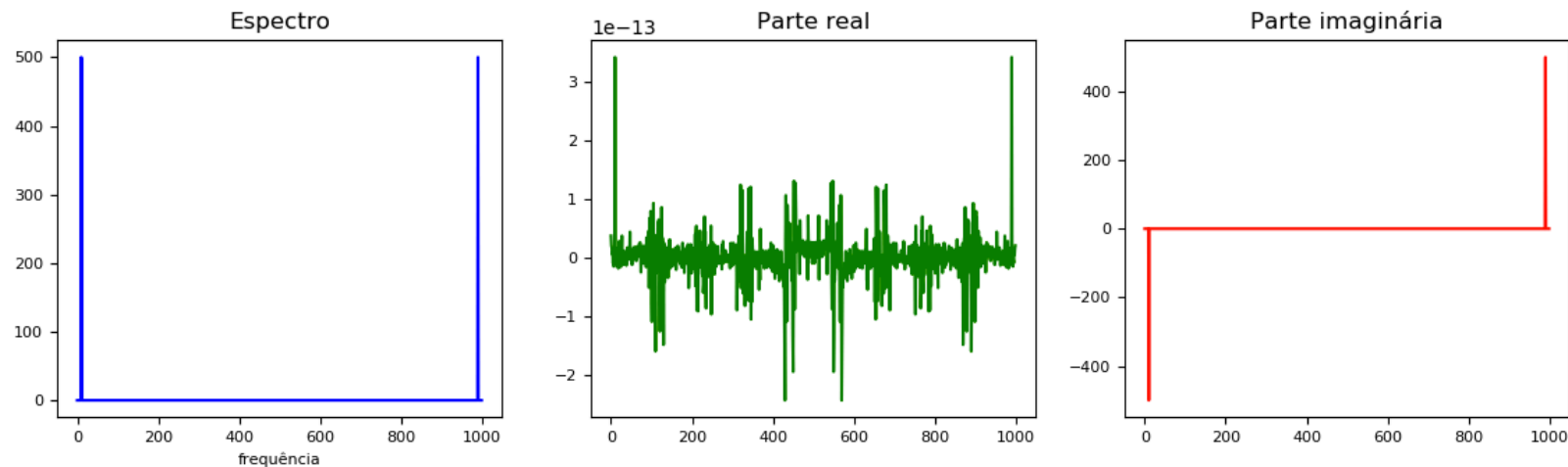
$f(x)$ = signal value at x

u = current frequency (0 Hertz to $N-1$ Hertz)

$F(u)$ = frequency amplitude u present in signal

1D Fourier Transform

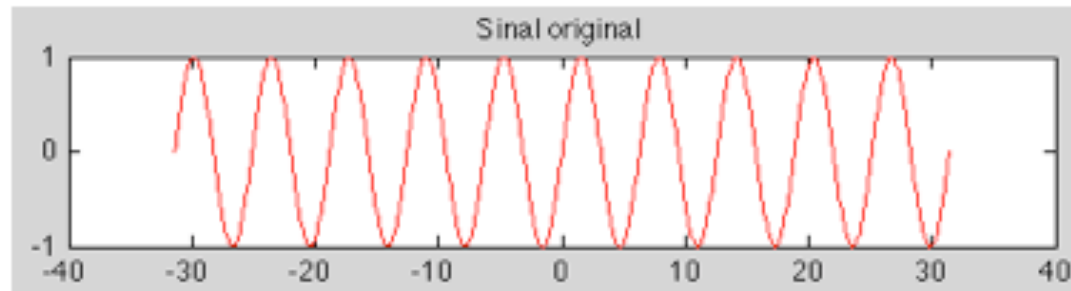
The frequency spectrum of a function results from the application of DFT. The values resulting from this transformation belong to the domain of complex numbers ($z = a + bj$), consisting of a real part (a) and an imaginary part (b). The magnitude of the spectrum is equal to the absolute value of DFT ($|z|$)



Magnitude

Frequency spectrum

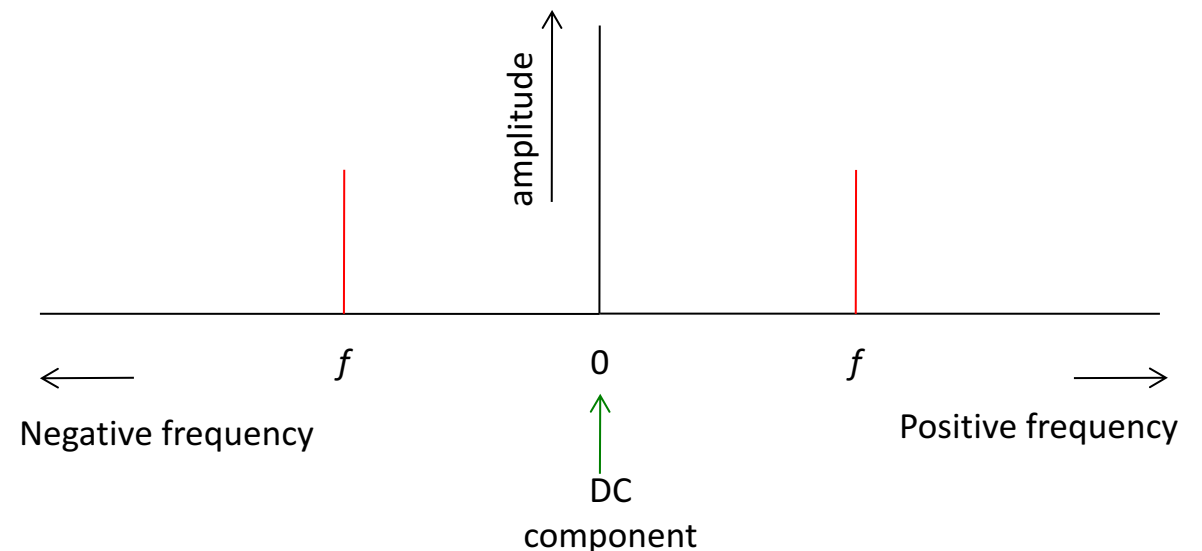
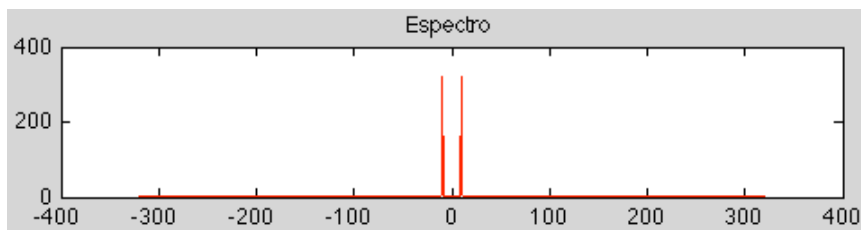
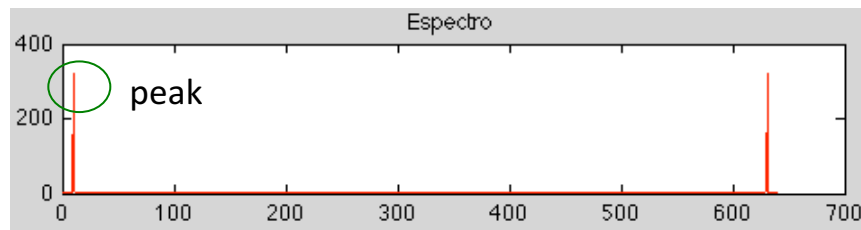
Consider the following periodic signal (sine).



The **spectrum** of a signal contains the magnitudes of the wave frequencies that make it up. In this example, since the signal represents a sinusoidal function (sine), there is only one frequency.

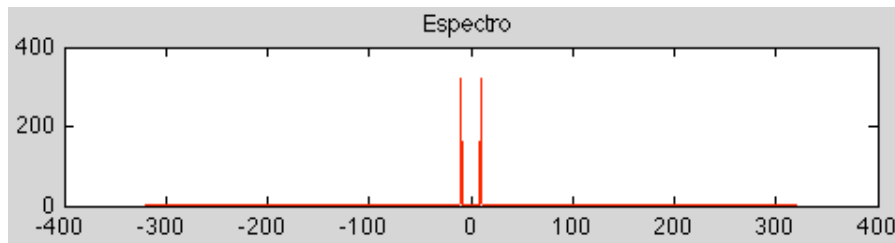
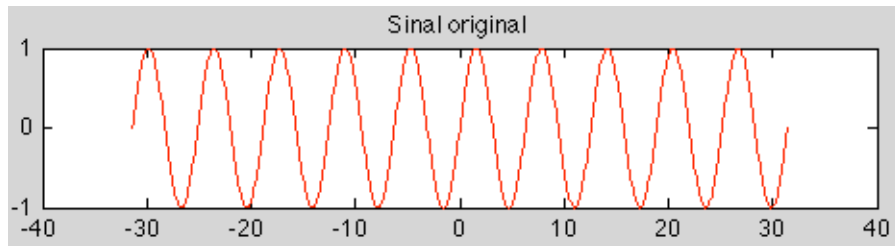
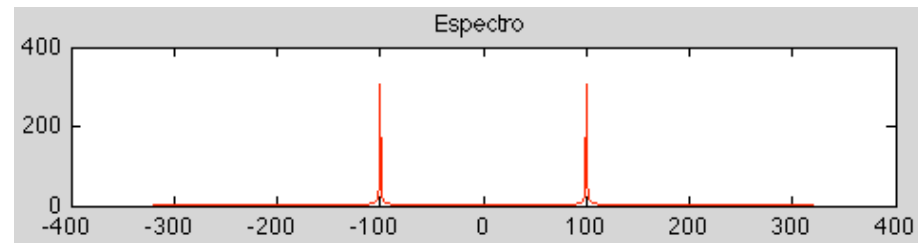
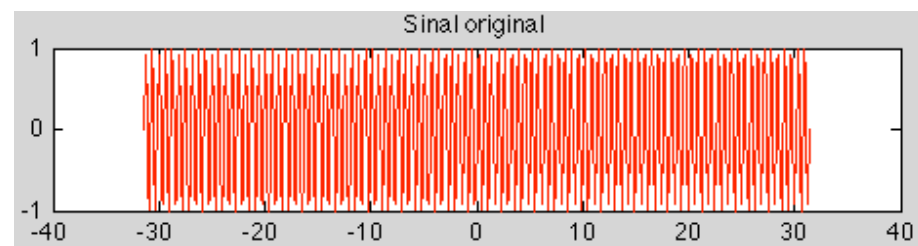
Frequency spectrum

The spectrum has two parts: a negative part and a positive part. The negative part contains negative frequencies. For real signals (without imaginary part), the negative part of the spectrum is always a “mirrored” version of the positive part. Thus, in this example, the positive part will have only one peak, and the negative part will have the same peak as the positive part.



Frequency spectrum

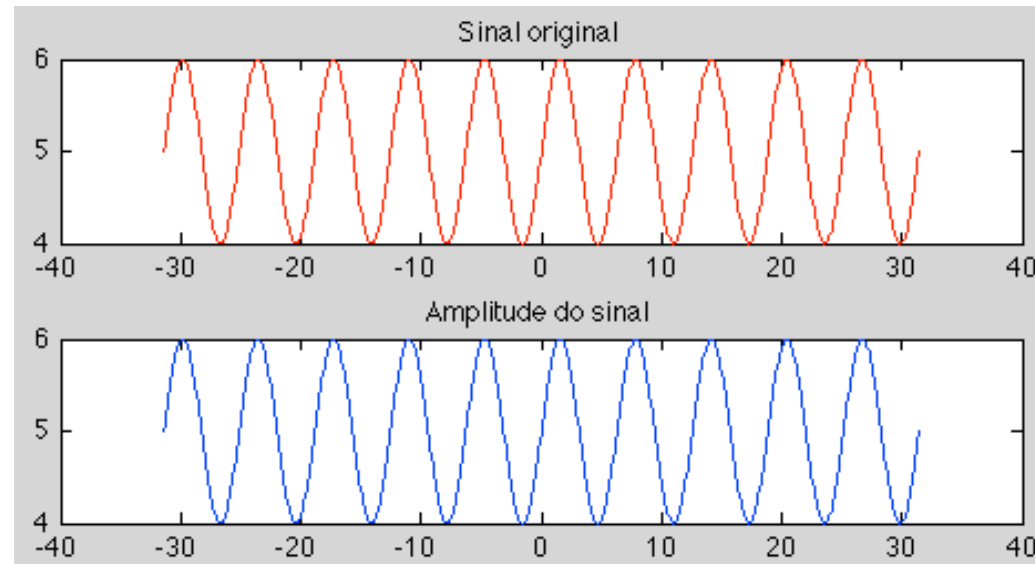
The farther to the left (and to the right) is a peak, the higher the frequency it represents. In other words, a peak far to the right (and left) means that the signal contains a periodic high frequency component.

 f  $10 \times f$ 

Frequency spectrum

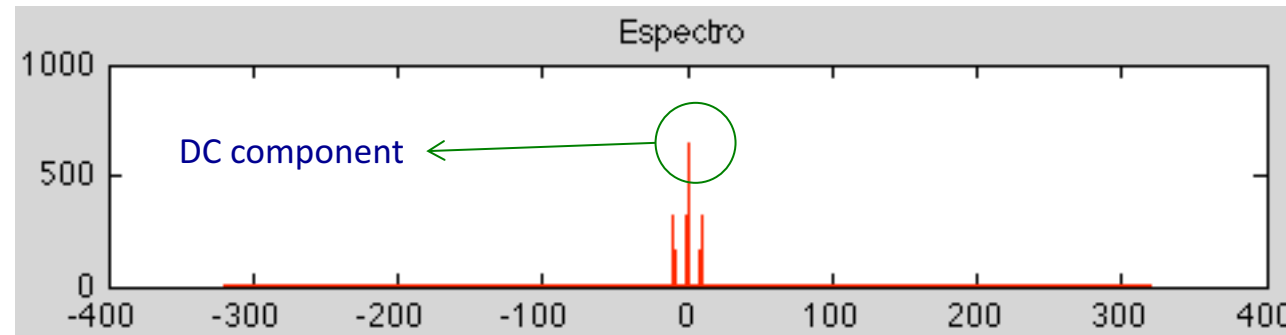
Consider a second example of a sinusoidal curve added by a value C : $f(x) = \sin(x) + C$. The action of this sum is to translate the signal in the yy direction. The mean value of the signal is equal to the value of C (since the mean value of $\sin(x) = 0$).

$C = 5$



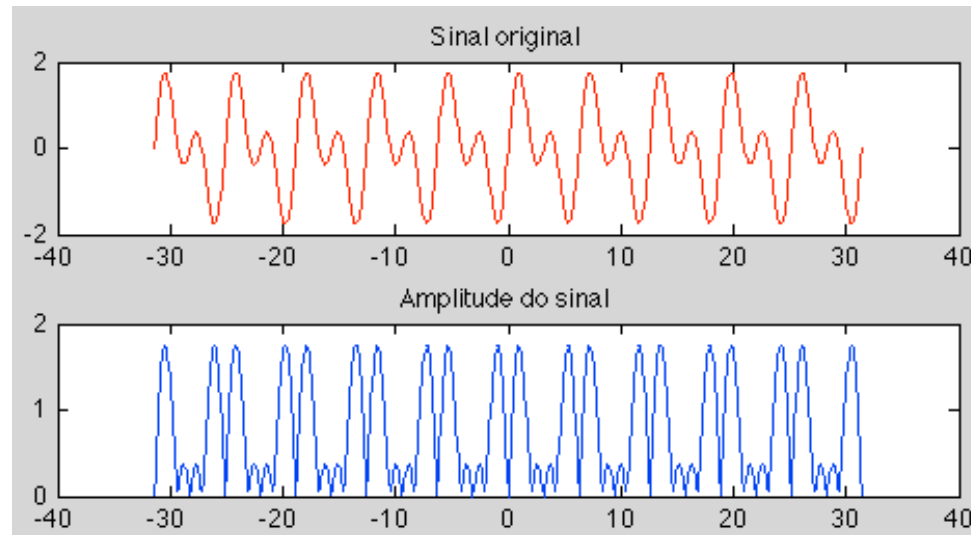
Frequency spectrum

In the frequency spectrum the frequency $f = 0$ corresponds to the DC component. This component has an amplitude equal to the average signal value. When a signal has a nonzero mean value, in addition to the frequency peaks of the sine wave components, the zero frequency of the spectrum ($f = 0$) has an additional peak which is called the DC component and has an amplitude equal to the mean value. Signal Thus, if the spectrum of a signal has a nonzero value at the source, it will be known that the average signal value is nonzero.



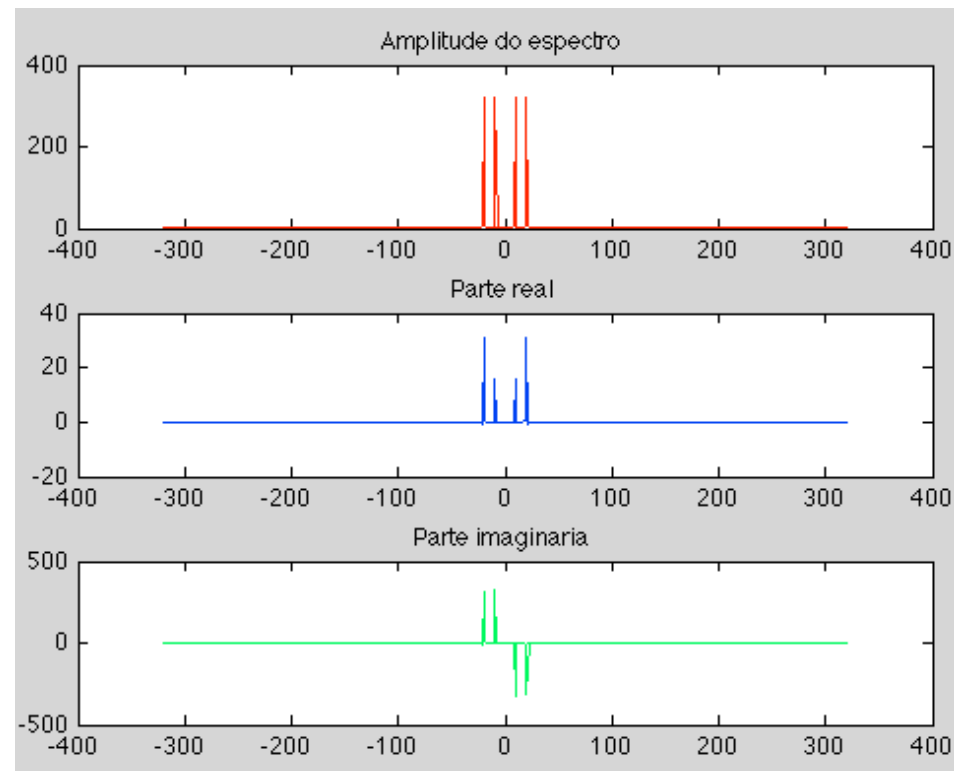
Frequency spectrum

Consider a third example, such as the sum of two sine functions, where the second has a double frequency of the first: $f(x) = \sin(kx) + \sin(2kx)$. The sign has the following form:



Frequency spectrum

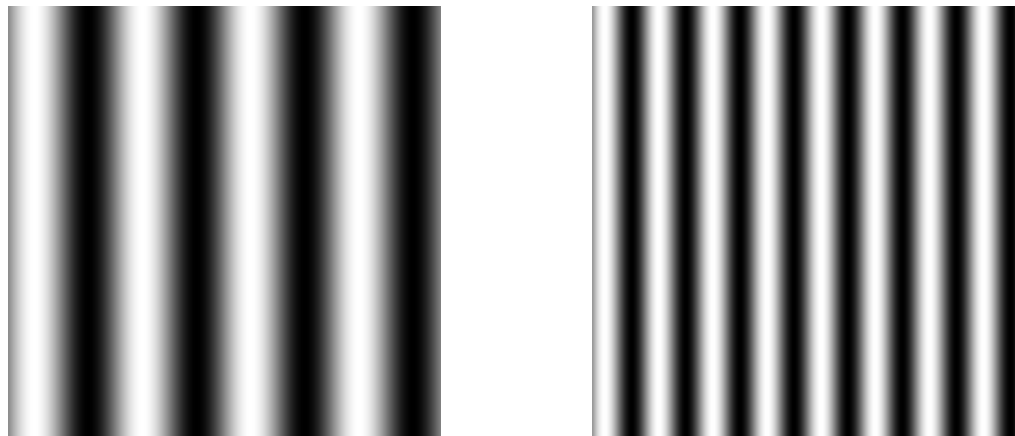
Since there are two sine functions with two different frequencies, two peaks can be expected on the positive side of the spectrum (and two on the negative side).



2D Fourier Transform

In a 2D signal (an image) the sinusoidal variations are represented by the gray scale variations of the pixels along the image.

The spatial frequency corresponds to the frequency along the space where there is intensity modulation (in the case of the images below, along the xx axis). The left image has a lower spatial frequency than the right one.



2D Fourier Transform

Transforming the spatial domain to the spatial frequency domain results in a function of complex values, that is, type $z = a + bj$.

$$F(u) = re(F(u)) + im(F(u))j$$

re: real part of the complex number

im: imaginary part of the complex number

$$\text{Magnitude} = |F(u)| = \sqrt{(re(F(u)))^2 + (im(F(u)))^2}$$

$$\text{phase} = atan\left(\frac{im(F(u))}{re(F(u))}\right)$$

Spectrum visualization can be done by determining the Magnitude function.

2D Fourier Transform

Discrete Fourier Transform (2D): For an image of dimensions $M \times N$, the two-dimensional Discrete Direct Fourier Transform is given by:

$$F(u, v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} \left[f(x, y) \times e^{-j2\pi \left(\frac{u \times x}{M} + \frac{v \times y}{N} \right)} \right]$$

The function $f(x, y)$ is the image in the spatial domain and the exponential term is the base function that corresponds to the representation of each point $F(u, v)$ of the Fourier space.

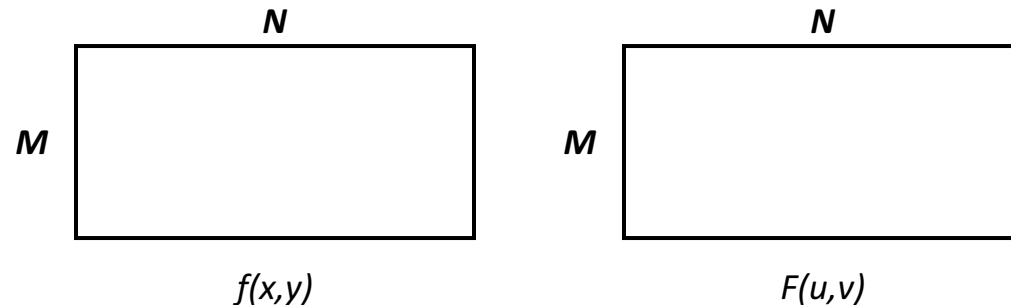
The magnitude of the spectrum is calculated with the following expression:

$$Mag(u, v) = \frac{1}{(N \times M)} |F(u, v)|$$

2D Fourier Transform

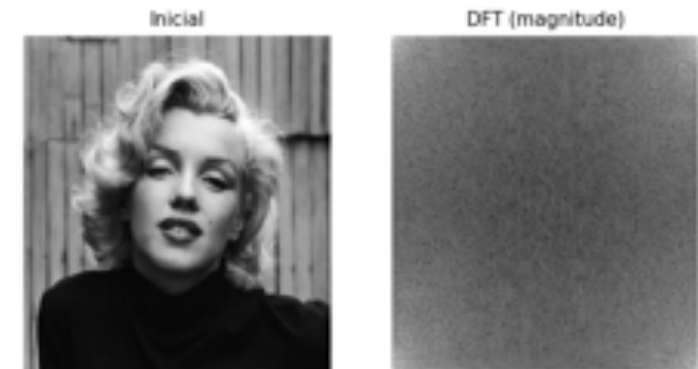
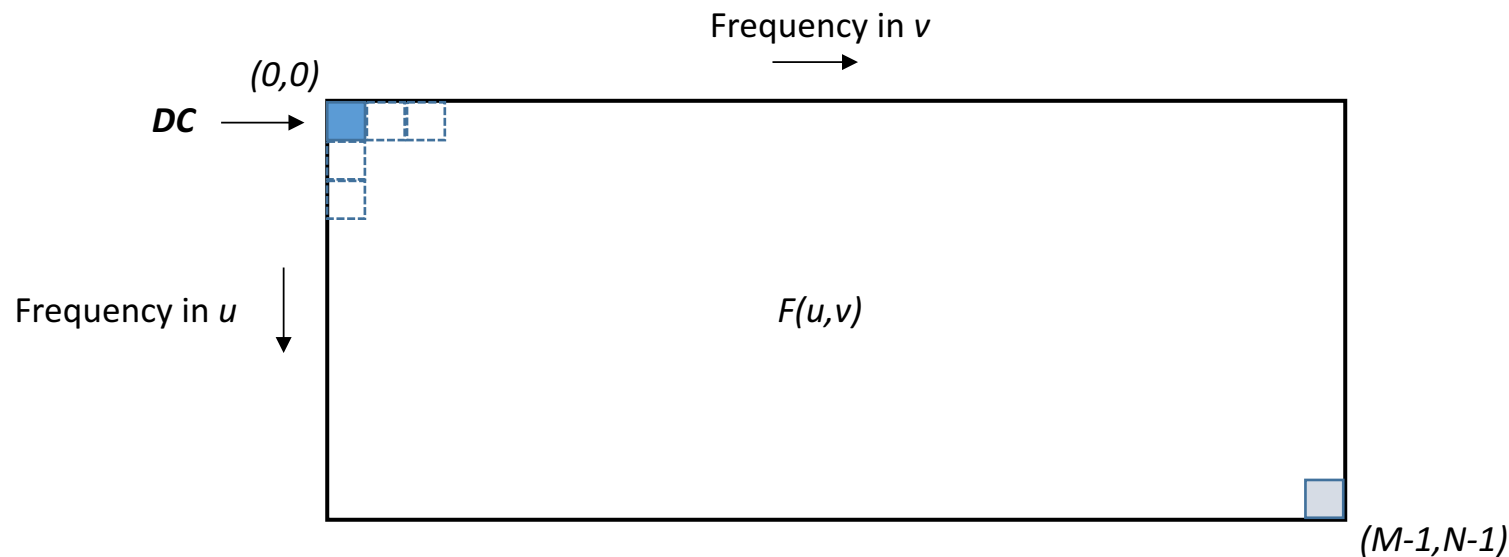
The Discrete Direct Fourier Transform (DDFT) corresponds to the sampled TF and as such does not contain all the frequencies that form an image, but only a set of samples that is large enough to describe the spatial domain of the image.

The number of frequencies corresponds to the number of pixels in the spatial domain image $f(x, y)$, that is, the image and spectrum have the same dimensions.



2D Fourier Transform

The base functions are sine and cosine waves with progressive frequencies, that is, $F(0, 0)$ represents the DC component of the image (which has zero frequency), corresponding to the average intensity, and $F(M-1, N-1)$ represents the highest frequency in xx and yy .



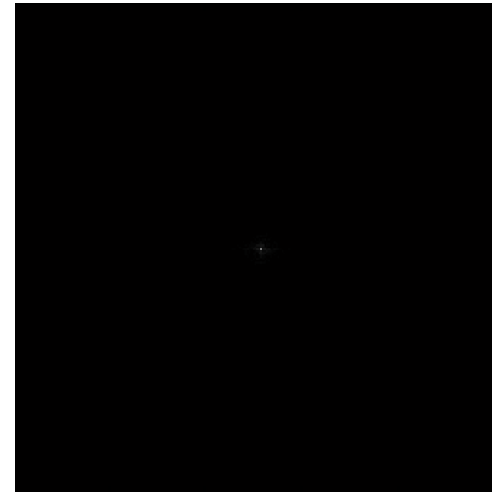
2D Fourier Transform

Almost always the DC value is by far the largest component of the frequency spectrum. The numerical range of the spectrum values is too large to be viewed on the screen, making their representation not often feasible.

Initial image



Magnitude spectrum



Min = 1.4514; Max = 6955093

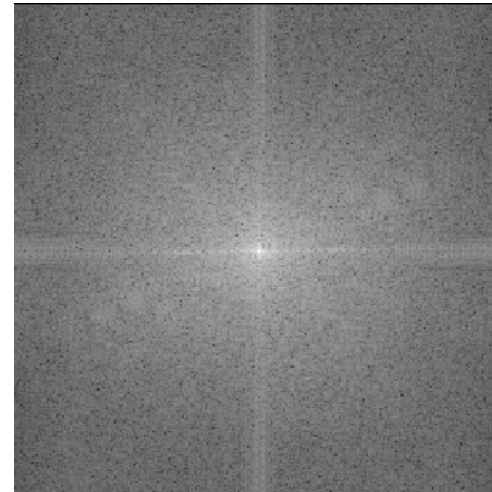
2D Fourier Transform

Applying the logarithmic operator to the Magnitude function for an 8-bit representation gives an appropriate representation of the frequency spectrum (alternatively, sometimes the magnitude matrix is divided by the total number of pixels).

Initial image



Magnitude spectrum

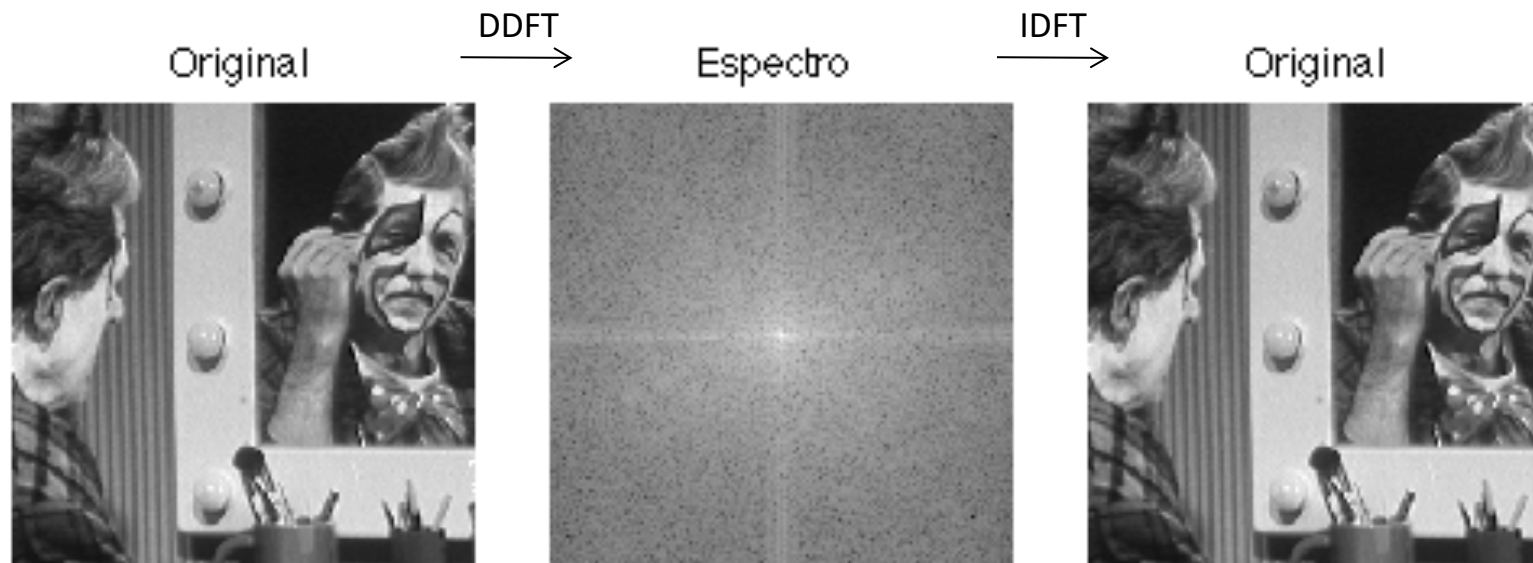


Min = 0; Max = 255

2D Fourier Transform

The two-dimensional Discrete Inverse Fourier Transform (IDFT) is given by:

$$f(x, y) = \left(\frac{1}{M \times N} \right) \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} \left[F(u, v) \times e^{j2\pi \left(\frac{u \times x}{M} + \frac{v \times y}{N} \right)} \right]$$



2D Fourier Transform

IDFT generally needs the real and imaginary parts obtained from DDFT.

Inicial



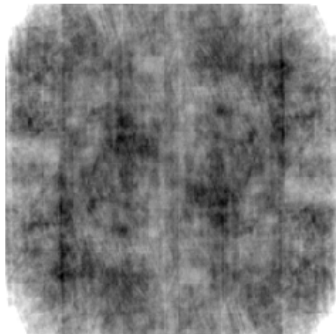
ITF da parte real



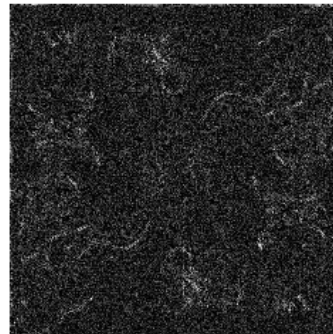
ITF da parte imaginaria



ITF do modulo

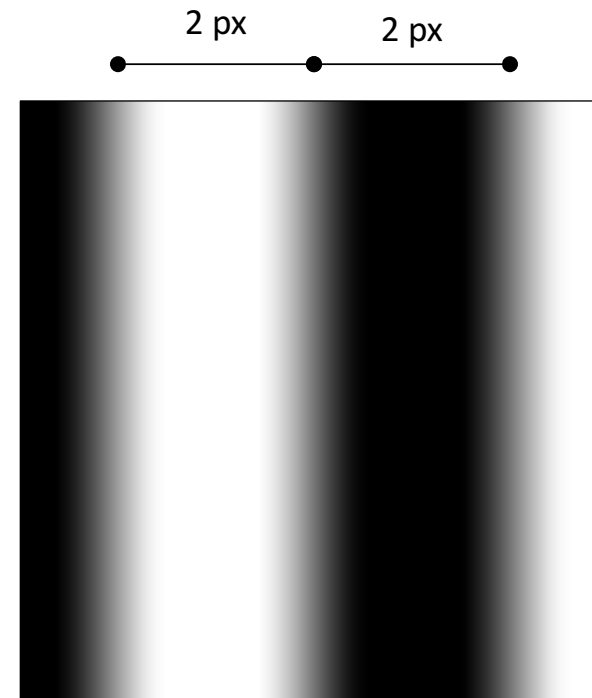
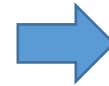
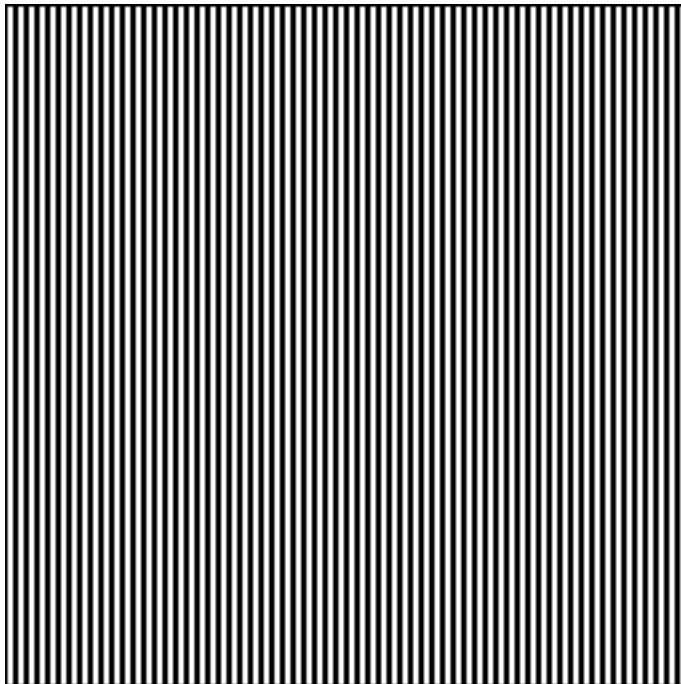


ITF da fase



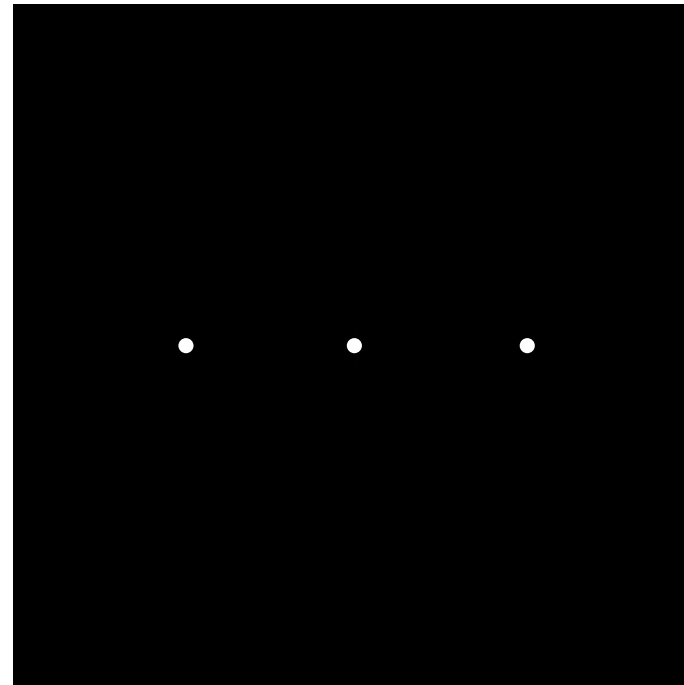
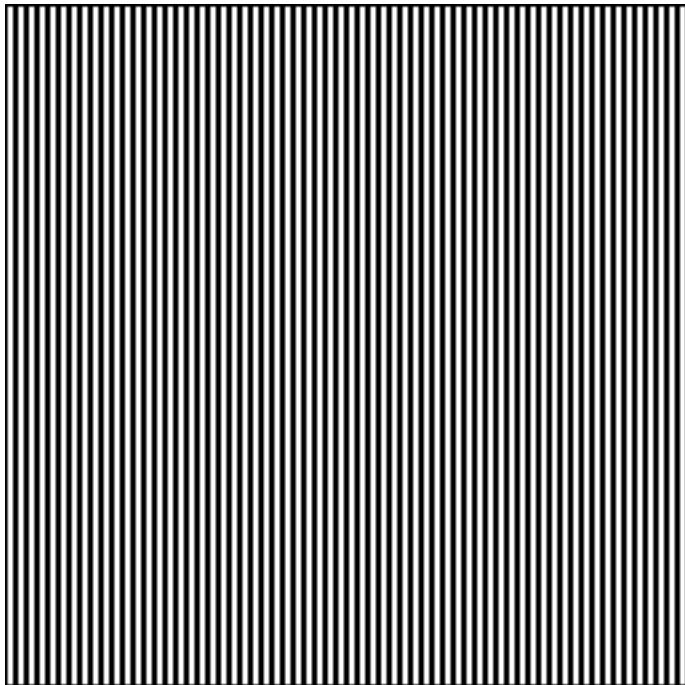
2D Fourier Transform

As an example, consider the image below, where each vertical strip (white or black) has a thickness of 2 pixels.



2D Fourier Transform

The Fourier transform results in the right spectrum that has three frequencies: the DC value and, as the spectrum is symmetrical with respect to the center, the two points corresponding to the frequency of the lines in the original image.





2D Fourier Transform

Since in the spatial domain the function changes in the horizontal direction, the points of the spectrum are aligned along a horizontal line passing through the center.

In the spectrum, the distance from the points to the center can be explained as follows: the maximum frequency (k_{\max}), which can be represented in the original image, is 2 pixels apart (one black and one white):

$$k_{\max} = \frac{1}{2}$$

2D Fourier Transform

The signal consisting of vertical bands with a thickness of 2 pixels (4 pixels cycle) has:

$$k = \frac{1}{4} = \frac{k_{\max}}{2}$$

Therefore, the points of the spectrum are midway between the center and the limits of the image, ie the frequency represented is half of the maximum frequency.



Filtering in frequency domain

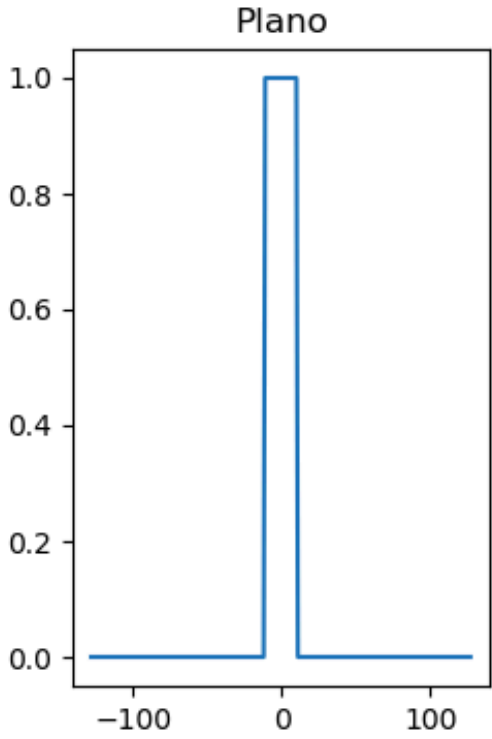
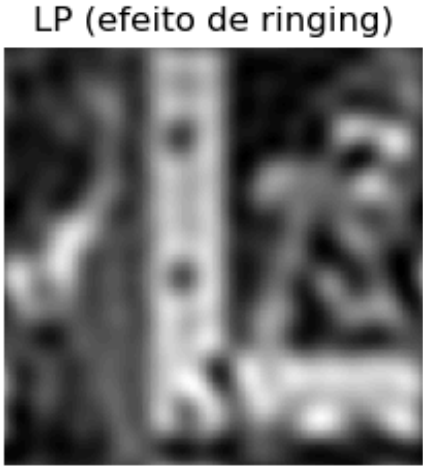
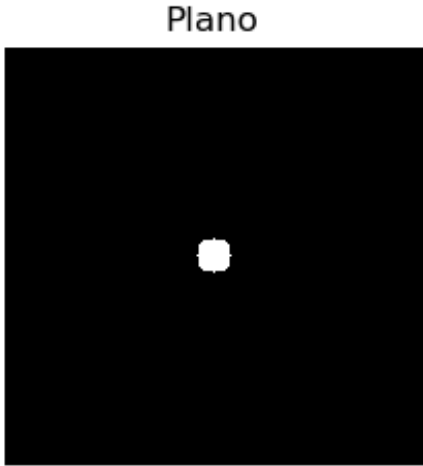
TF is directly related to many operations, including filtering operations.

A filtering operation, which in the spatial domain results from a convolution operation between a filter (kernel) and the gray image, is performed with a simple multiplication in the frequency domain (TF of the initial image).

Thus low pass, high pass and band pass filters can be defined, for example by eliminating spectrum frequencies.

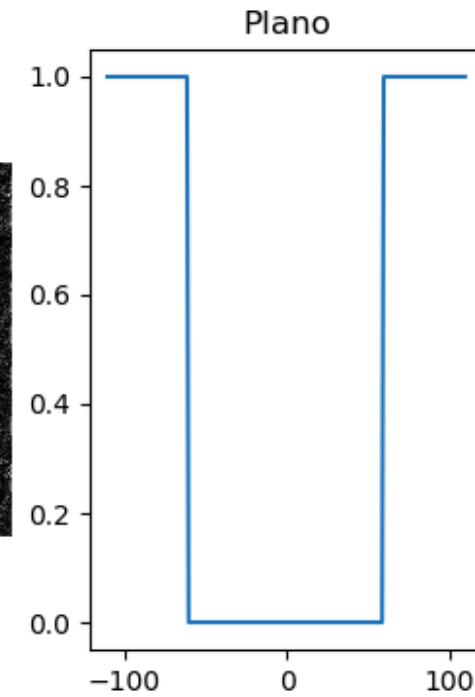
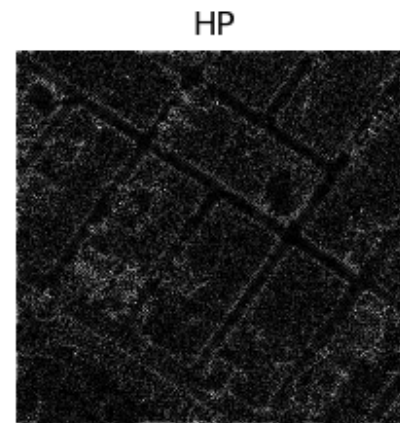
Filtering in frequency domain

Flat Low Pass filter:



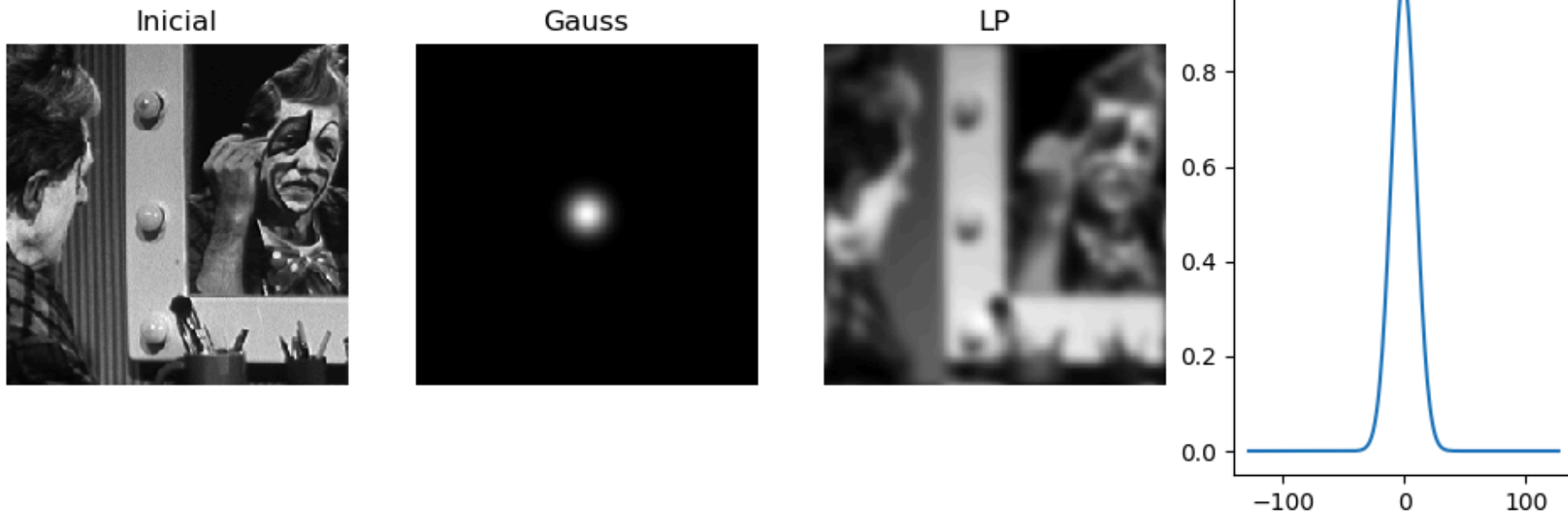
Filtering in frequency domain

Flat High Pass filter:



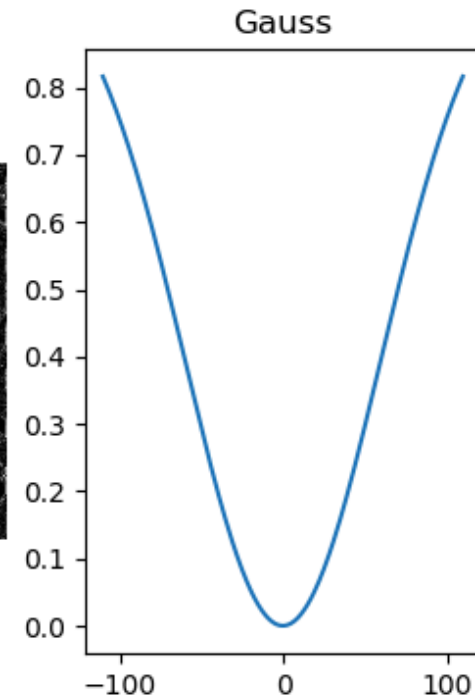
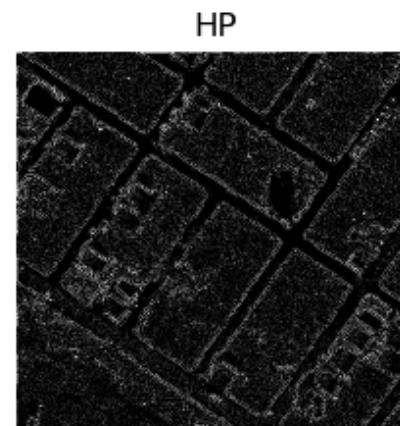
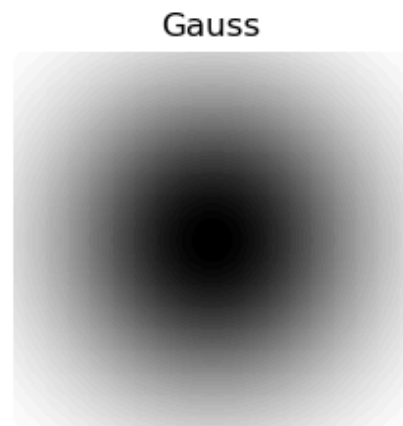
Filtering in frequency domain

Gaussian Low Pass filter



Filtering in frequency domain

Gaussian High Pass filter



Filtering in frequency domain

The **Butterworth** filter of order n and cutoff frequency $D0$ is defined as:

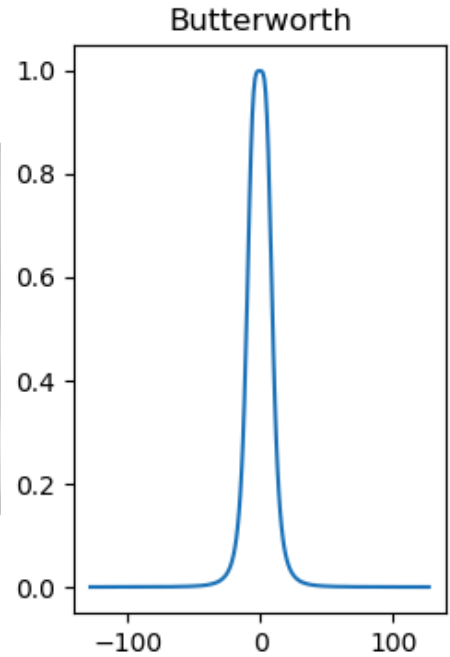
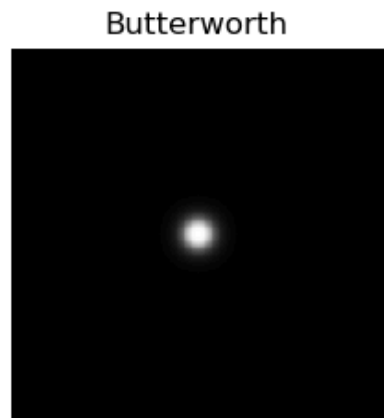
$$B(u, v) = \frac{1}{\left[1 + \left(\frac{\sqrt{u^2 + v^2}}{D0}\right)^{2n}\right]}$$

This filter has the advantage that you can control the sharpness with the order (n).

A Butterworth lowpass filter maintains frequencies within radius $D0$ and discards those outside. It also introduces a 1 to 0 gradual transition to reduce the ringing effect of the flat filter.

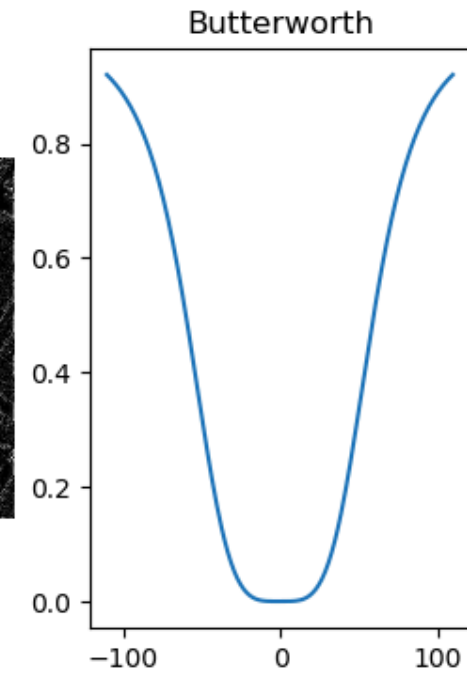
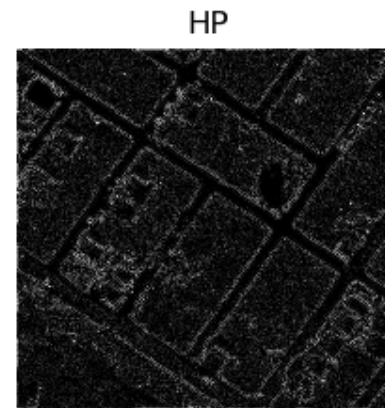
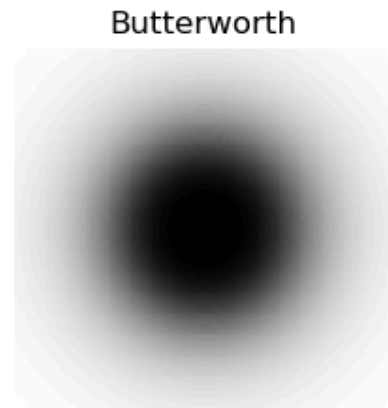
Filtering in frequency domain

Butterworth Low Pass filter



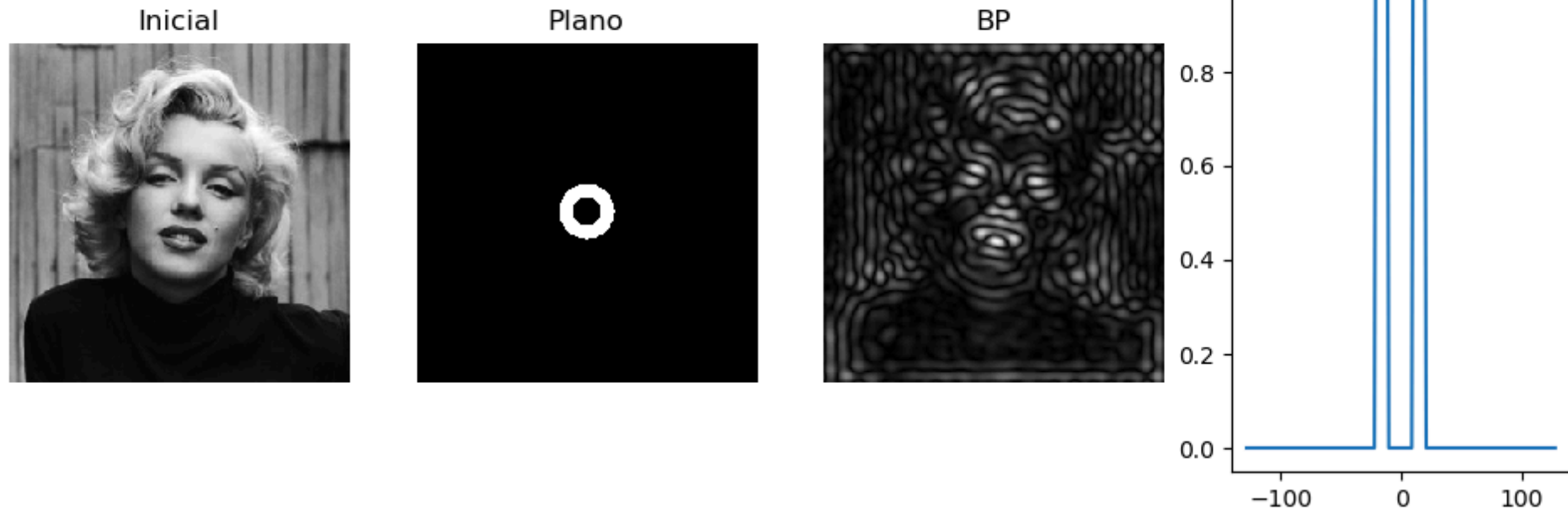
Filtering in frequency domain

Butterworth High Pass filter



Filtering in frequency domain

Flat Band Pass filter



Convolution theorem

In mathematics, the convolution theorem states that, under suitable conditions, the TF of a two-signal convolution is obtained by multiplying its individual Fourier transforms and performing the inverse of the TF of that product.

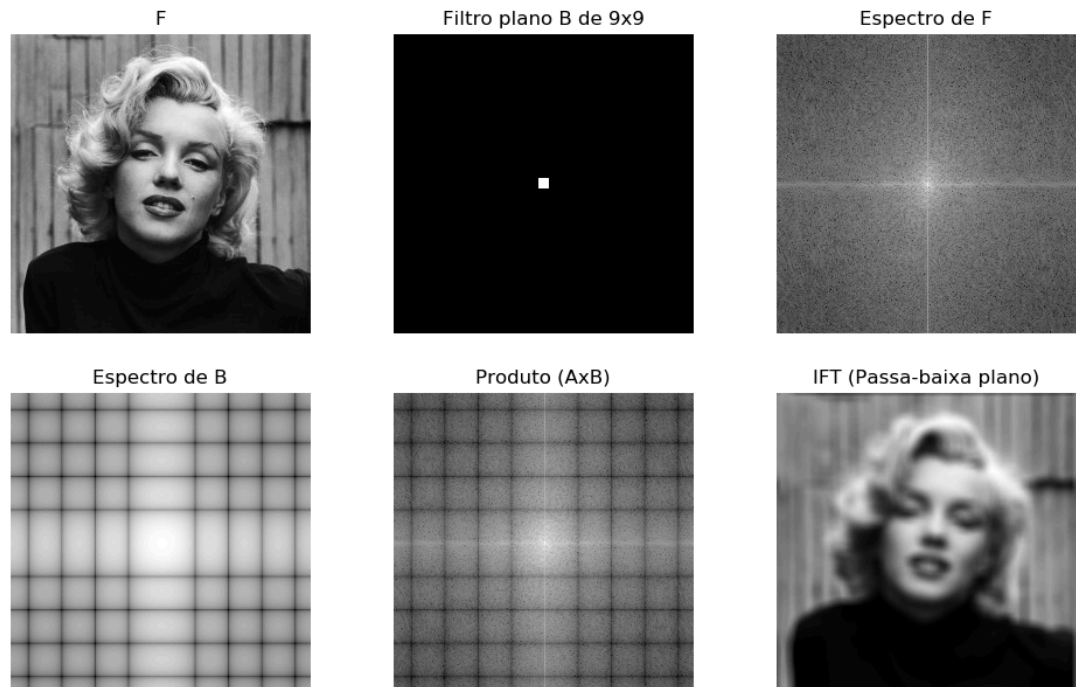
$$f(x, y) * h(x, y) = idft(F(u, v) \times H(u, v))$$

This theorem is important because it establishes the link between frequency domain operations and the action of linear spatial filters.

Linear filtering operations performed by convolution in the spatial domain can be performed by simple multiplications in the Fourier domain, making the filtering process faster.

Convolution theorem

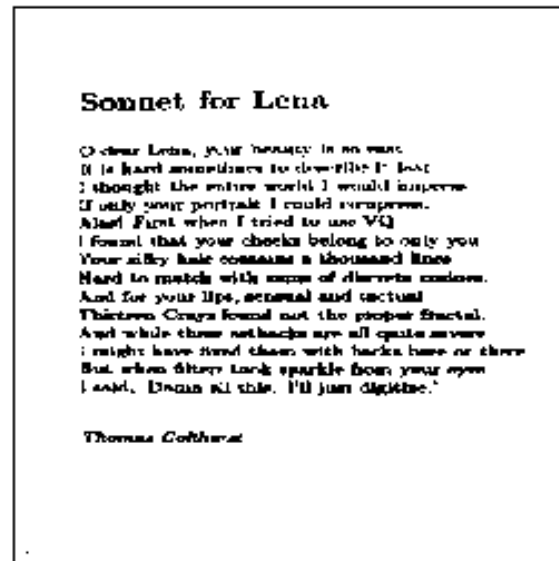
To do this, simply multiply the desired filter matrix by the resulting DFT image (spectrum with both real and imaginary parts) and then apply the IFT. In the figure one can see the illustration of the filtering process of an image by a rectangular filter of the arithmetic mean.



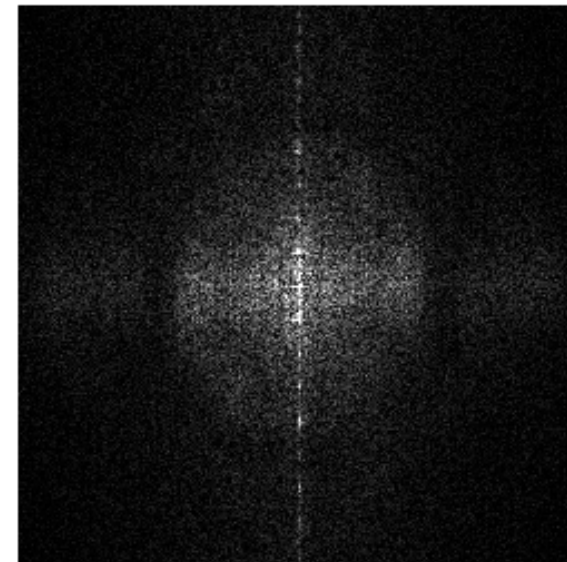
Application Examples

Text Orientation Detection: TF is used to acquire information about the geometric structure of an image's spatial domain. Text recognition using image processing techniques is simplified if it is assumed that the lines of a given text are arranged in a given direction.

Original



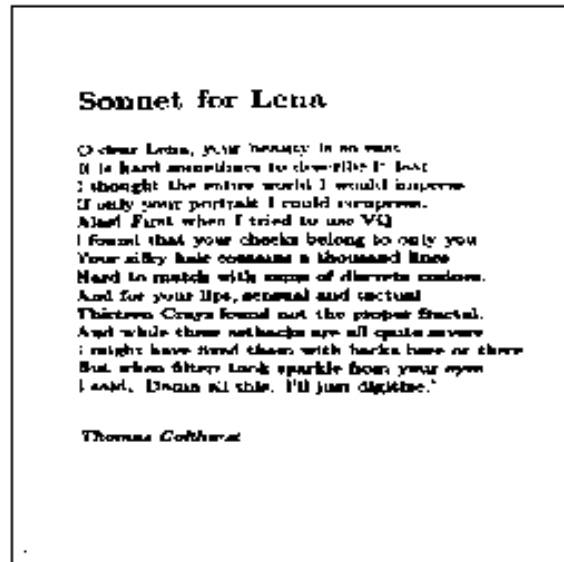
Espectro linxcol



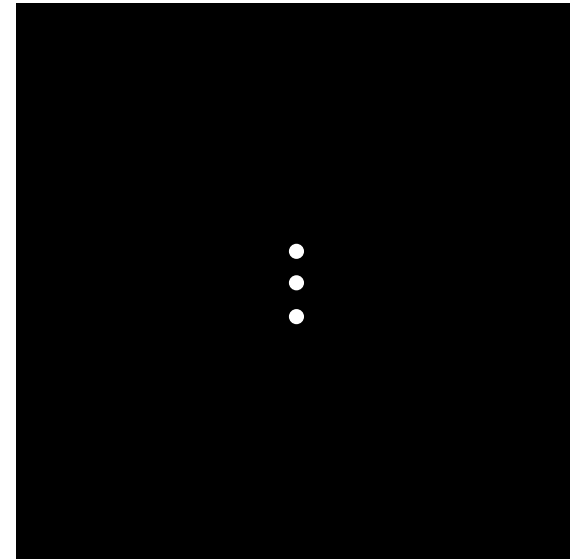
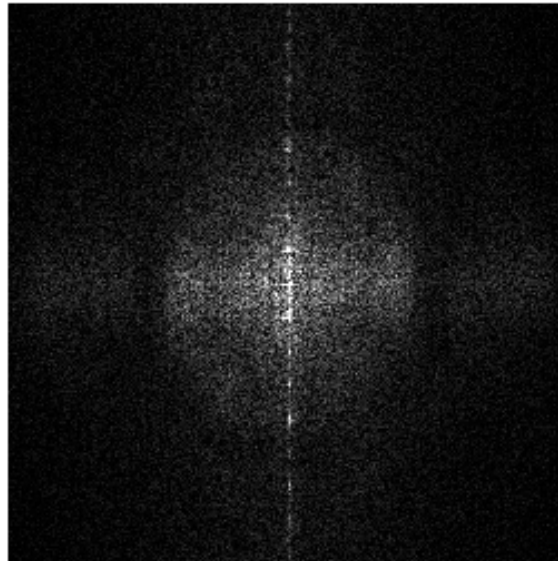
Application Examples

Spectrum image peak thresholding allows to identify the orientation of the text line layout.

Original

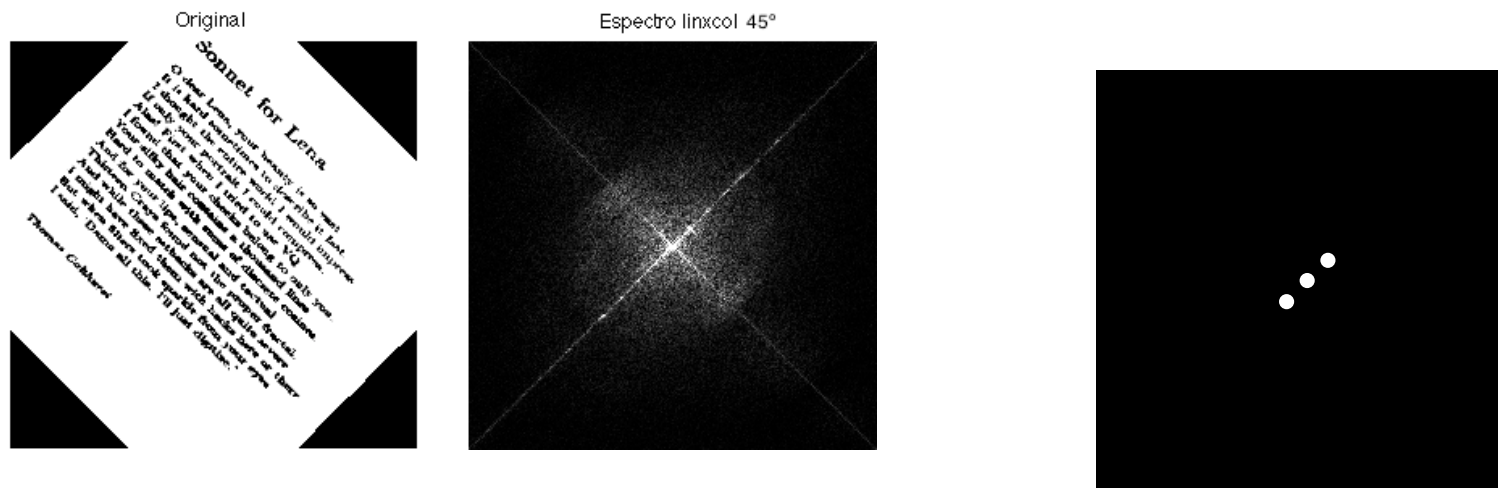


Espectro linxcol



Application Examples

In the case of a 45° rotated image, after identifying the frequency peaks on the main line, the image can be rotated using the knowledge of the angle obtained in this procedure.



The line perpendicular to the main line results from dark corners in the rotated image.

Application Examples

The Fourier Transform can be used to perform operations of locating objects in an image by determining the cross power spectrum (which will be seen in the image segmentation theme).

